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RISK SHARING IN LIFE INSURANCE AND PENSIONS

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BACKGROUND IN KEYWORDS

Traditional insurance mathematics: Perfect internal management of demographic and financial risk through with-profit schemes adapting payments to indices for interest, mortality, etc.

Deregulation in the 80-es lead to:

In life insurance and pensions: with profit gave way to a plethora of products equipped with various forms of guarantees (risk creation). Risk management increasingly based on market operations: reinsurance, swaps, securitization. Overreliance in the efficient market hypothesis (on which modern financial mathematics builds).

Parallel development in actuarial science seen as an area of application of financial maths – pricing and hedging of guarantees.

Pension crisis triggered by faltering markets and improved longevity and by the guarantee culture.

Reintroduction of regulation. Solvency II, IFRS.

Renewed interest in with profit? Or seek new ways of adapting premiums and benefits to indices for interest and mortality?
SAVINGS CONTRACT

$S_j$ price of one unit of asset at time $j = 1, 2, ...$

$c_j$ invested at time $j = 1, 2, ..., m$

Contribution at time $j$ purchases $u_j = \frac{1}{S_j} c_j$ units ($c_j = u_j S_j$)

Value of portfolio at time $t = 1, 2, ..., m$ is

$$V_t = S_t \sum_{j=1}^{t} u_j = S_t \sum_{j=1}^{t} \frac{1}{S_j} c_j$$

$b_j$ benefit withdrawn at time $j = m + 1, ..., T$

Value of portfolio at time $t = m + 1, ..., T$ is

$$V_t = S_t \left( \sum_{j=1}^{m} \frac{1}{S_j} c_j - \sum_{j=m+1}^{t} \frac{1}{S_j} b_j \right)$$
Contract terminates at time \( T \) with balance \( V_T = 0 \):

\[
\sum_{j=1}^{m} \frac{1}{S_j} c_j = \sum_{j=m+1}^{T} \frac{1}{S_j} b_j
\]

Level contributions \( c_j = c \) and benefits \( b_j = b \):

\[
\frac{c}{b} = \frac{\sum_{j=m+1}^{T} \frac{1}{S_j}}{\sum_{j=1}^{m} \frac{1}{S_j}}
\]

If \( S_j \) increasing then \( \frac{c}{b} < \frac{T-m}{m} \).

The “steeper” \( S_j \), the smaller \( c/b \).
MORTALITY

Large number $\ell_0$ of new-born
$
\ell_x$ survivors at age $x = 1, 2, \ldots$

deaths at age $x$
$mortality rate at age $x$

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PENSIONS CONTRACT

\[ V_T = S_T \left( \sum_{j=1}^{m} \frac{1}{S_j} \ell_{x+j} c_j - \sum_{j=m+1}^{T} \frac{1}{S_j} \ell_{x+j} b_j \right) \]

\( V_t \geq 0 \) for all \( t \) and \( V_T = 0 \):

\[ \sum_{j=1}^{m} \frac{1}{S_j} \ell_{x+j} c_j = \sum_{j=m+1}^{T} \frac{1}{S_j} \ell_{x+j} b_j \]

Level contributions \( c_j = c \) and benefits \( b_j = b \):

\[ \frac{c}{b} = \frac{\sum_{j=m+1}^{T} \frac{1}{S_j} \ell_{x+j}}{\sum_{j=1}^{m} \frac{1}{S_j} \ell_{x+j}} \]
NUMERICAL ILLUSTRATIONS: $x = 30$, $m = 35$, $T = 70$, values of $c/b$:

No interest, no mortality: 1.0000
4% interest, no mortality: 0.2538
4% interest, mortality as in table (mean life length 73): 0.1149
4% interest, mortality half of table (mean life length 81): 0.1592
2% interest, mortality as in table: 0.2035
2% interest, mortality half of table: 0.2918

$m = 39$, $T = 70$
2% interest, mortality as in table: 0.2101
RISK MANAGEMENT. Balance requirement (the principle of equivalence):

\[
\sum_{j=1}^{m} \frac{1}{S_j} \ell_{x+j} c_j = \sum_{j=m+1}^{T} \frac{1}{S_j} \ell_{x+j} b_j
\]  

(1)

If future indices \( S_j \) and \( \ell_{x+j} \) are uncertain, equivalence can only be attained if the payments are allowed to depend on the indices.

We shall consider two ways of doing this.
PERFECT UNIT LINKED INSURANCE (index-regulated payments): At time 0 specify a technical basis $S_j^*$ and $\ell_j^*$ representing “best estimates” of future indices, and baseline payments $c_j^*$ and $b_j^*$ representing intended profile of the payments, designed to satisfy equivalence under the technical scenario:

$$\sum_{j=1}^{m} \frac{1}{S_j^*} \ell_x+j c_j^* = \sum_{j=m+1}^{T} \frac{1}{S_j^*} \ell_x+j b_j^*$$

Actual payments are index-regulated:

$$c_j = \frac{S_j \ell_x+j}{S_j^* \ell_x+j} c_j^* \quad b_j = \frac{S_j \ell_x+j}{S_j^* \ell_x+j} b_j^*$$

Equivalence (1) is then attained under all scenarios.
Solvency for sure.

No assumptions about future interest and mortality needed

Unit linked in its pure form is not used in practice. Usually premiums are not index linked and benefits are equipped with a guarantee that they cannot go below a certain nominal level.

Such guarantees reintroduce risk on the part of the insurer.
WITH PROFIT INSURANCE

Contractual contributions $c_j^*$ and benefits $b_j^*$ specified at time 0 such that reserve is non-negative and balance is attained under prudent \textit{technical} assumptions ($S_0^* = 1$), $S_j^*$, $\ell_{x+j}^*$, $j = 1, \ldots, T$:

$$V_t^* \geq 0, \ t = 1, \ldots, T - 1$$
$$V_T^* = 0$$ (2)

Technical reserve at time $t$ (retrospective = prospective due to (2)):

$$V_t^* = S_t^* \sum_{j=1}^{t} \frac{1}{S_j^*} \ell_{x+j}^* (c_j^* - b_j^*) = S_t^* \sum_{j=t+1}^{T} \frac{1}{S_j^*} \ell_{x+j}^* (b_j^* - c_j^*)$$ (3)

(Notation: $b_j^* = 0$, $j = 1, \ldots, m$, $c_j^* = 0$, $j = m + 1, \ldots, T$.)
Factual interest and mortality are given by \((S_0 = 1), S_j, \ell_{x+j}, j = 1, \ldots, T\). Non-negative dividends (bonuses) \(d_j, j = 1, \ldots, T\), are paid if technical assumptions generate surplus. Define \textit{surplus} at time \(t\) as accumulated value of past payments less discounted value of future payments under technical assumptions:

\[
W_t = \sum_{j=1}^{t} \frac{S_t}{S_j} \ell_{x+j} (c_j^* - b_j^* - d_j) - \ell_{x+t} \sum_{j=t+1}^{T} \frac{S_t^*}{S_j^*} \frac{\ell_{x+j}^*}{\ell_{x+t}^*} (b_j^* - c_j^*)
\]

Use (3) and rearrange to write \(W_t\) as technical surplus less dividends:

\[
W_t = \sum_{j=1}^{t} (1 - \delta_{t,j}) \frac{S_t}{S_j} \ell_{x+j} (b_j^* - c_j^*) - \sum_{j=1}^{t} \frac{S_t}{S_j} \ell_{x+j} d_j \tag{4}
\]

\[
\delta_{t,j} = \frac{S_t^*/S_j^*}{S_t/S_j} \frac{\ell_{x+t}/\ell_{x+j}}{\ell_{x+t}^*/\ell_{x+j}^*} \leq 1 \text{ if } S_t^*/S_j^* \leq S_t/S_j \text{ and } \ell_{x+t}^*/\ell_{x+j}^* \geq \ell_{x+t}/\ell_{x+j}
\]
Pay back surplus in arrears as bonus (not a contractual obligation) in a manner that balance is attained at time $T$. Solvency ensured if technical assumptions are sufficiently prudent.

**No assumptions about future interest and mortality needed**

The contractual payments are binding to the insurer throughout the term of the contract. Thus, the technical basis represents a guarantee to the policy holder that benefits will not be less than stipulated in the contract (and premiums cannot be raised) no matter what future mortality and investment experience will be. Such guarantees represent risk on the part of the insurer.
MARKET METHODS

MORTALITY SWAPS:
Essentially a form of reinsurance. OTC - not liquidly traded. No price record.

MORTALITY DERIVATIVES:
Can work for catastrophic mortality risk in the short term. Risk is limited in time and space and can be absorbed by capital markets. Catastrophes can be predicted by mathematical models fitted to historical data.
**Longevity risk** is all different:
Longevity derivatives need to be exceedingly long term. Longevity is the resultant of complex demographic, societal, and technological processes. Cannot be reliably predicted by mathematical models fitted to historical data. (Mathematical hedging theories are out.)
In the shorter term longevity developments can be anticipated from current collateral information to which investors will not have equal access. Arbitrages to be fetched by big investors/speculators.
The scale is too large. Pensions are a major accumulator of savings (bigger than the mortgage market), and longevity risk is at macroeconomic scale.
The very idea of longevity guarantees is questionable.
FINANCIAL MATHEMATICS

Traded assets:
Bank account with price process \( B_t = e^{rt} \)
Stock with price process \( S_t = e^{\alpha t + \sigma \sum_{i=1}^{t} X_i} \)

(Physical) Probability measure \( \mathbb{P} \): the \( X_i \) i.i.d. Bin(0, \( p \)), \( 0 < r - \alpha < \sigma \)

Portfolio at time \( t \) is \( (\eta_t, \xi_t) \) chosen at time \( t - 1 \)
Its value at time \( t \) is \( V_t = \eta_t B_t + \xi_t S_t \)
It is self-financing (SF) if
\[
V_t = \eta_{t+1} B_t + \xi_{t+1} S_t \quad \text{or} \quad (\eta_{t+1} - \eta_t) B_t + (\xi_{t+1} - \xi_t) S_t = 0
\]
Since \( V_{t+1} = \eta_{t+1} B_{t+1} + \xi_{t+1} S_{t+1} \), SF means
\[
\Delta V_t = V_{t+1} - V_t = \eta_{t+1} \Delta B_t + \xi_{t+1} \Delta S_t
\]

An SF portfolio is an arbitrage if \( V_0 < 0 \) and \( V_T \geq 0 \)
There should be no arbitrage in an efficient market (model!).
Discounted values $\tilde{B} = D_t B_t$, $\tilde{S} = D_t S_t$, $\tilde{V} = D_t V_t$

SF property is preserved under discounting:

$\tilde{V}_t = \eta_{t+1} \tilde{B}_t + \xi_{t+1} \tilde{S}_t$

Convenient to take $D_t = B_t^{-1}$. Then

$\tilde{B}_t = 1, \quad \tilde{S}_t = e^{(\alpha - r)t + \sigma \sum_{i=1}^{t} X_i}$,

\[
\begin{align*}
\Delta \tilde{B}_t &= 0 \\
\Delta \tilde{S}_t &= \tilde{S}_t \left[ X_{t+1}(e^{\alpha - r + \sigma} - 1) + (1 - X_{t+1})(e^{\alpha - r} - 1) \right] \\
&= \tilde{S}_t e^{\alpha - r}(e^\sigma - 1)(X_{t+1} - q) \\
\Delta \tilde{V}_t &= \xi_{t+1} \Delta \tilde{S}_t \text{ for SF portfolio}
\end{align*}
\]

$q = (e^{r - \alpha} - 1)/(e^\sigma - 1) \in (0, 1)$
Equivalent Martingale Measure (EMM) \( \mathbb{Q} \): the \( X_i \) i.i.d. Bin(0, \( q \)).

\( \tilde{S} \) is \( \mathbb{Q} \)-martingale: \( \mathbb{E}^{\mathbb{Q}}[\Delta \tilde{S}_t | \mathcal{F}_t] = 0 \).

\( \tilde{V} \) is \( \mathbb{Q} \)-martingale for SF portfolio: \( \mathbb{E}^{\mathbb{Q}}[\Delta \tilde{V}_t | \mathcal{F}_t] = \xi_{t+1} \mathbb{E}^{\mathbb{Q}}[\Delta \tilde{S}_t | \mathcal{F}_t] = 0 \)

Therefore, \( \mathbb{E}^{\mathbb{Q}} \tilde{V}_T = \tilde{V}_0 \), so there exists no arbitrage.

Suppose a traded derivative of the stock pays \( h(S_T) \) at time \( T \).

It is attainable if there exists SF portfolio with \( \mathbb{P}[V_T = h(S_T)] = 1 \).

Then the price of the derivative at time \( t \leq T \) must be \( V_t \).

The present market is complete: every financial derivative is attainable. Prove this by constructing SF portfolio such that \( \tilde{V}_T = e^{-rT}h(S_T) \) a.s.
Start from martingale $\tilde{V}_t = \mathbb{E}^Q[e^{-rT}h(S_T)|\mathcal{F}_t] = \mathbb{E}^Q[H(X_1, \ldots, X_T)|\mathcal{F}_t]$

Using independence, and writing $\mathbb{E}^*$ for $\mathbb{E}^Q$ w.r.t. $X_{t+2}, \ldots, X_T$ only,

$$\Delta \tilde{V}_t = \mathbb{E}^Q[H(X_1, \ldots, X_T)|\mathcal{F}_{t+1}] - \mathbb{E}^Q[H(X_1, \ldots, X_T)|\mathcal{F}_t]$$

$$= X_{t+1} \mathbb{E}^*H(X_1, \ldots, X_t, 1, X_{t+2}, \ldots, X_T)$$

$$+ (1 - X_{t+1}) \mathbb{E}^*H(X_1, \ldots, X_t, 0, X_{t+2}, \ldots, X_T)$$

$$- q \mathbb{E}^*H(X_1, \ldots, X_t, 1, X_{t+2}, \ldots, X_T)$$

$$- (1 - q) \mathbb{E}^*H(X_1, \ldots, X_t, 0, X_{t+2}, \ldots, X_T)$$

$$= \left[ \mathbb{E}^*H(X_1, \ldots, X_t, 1, X_{t+2}, \ldots, X_T) - \mathbb{E}^*H(X_1, \ldots, X_t, 0, X_{t+2}, \ldots, X_T) \right] (X_{t+1} - q)$$

$$= \xi_{t+1} \Delta \tilde{S}_t$$

Thus, $(\eta_t, \xi_t)_{t=0,\ldots,T}$ with $\eta_t = V_t - \xi_t S_t$ is SF portfolio attaining the claim.
LIFE INSURANCE AND PENSIONS

A \( T \)-year endowment with sum \( b \) is sold to an \( x \) year old against single premium \( \pi \).
Remaining life length \( T_x \), \( \mathbb{P}[T_x > t] = \exp[-\int_0^t \mu_{x+u} \, du] \)

Ulpianus (about 40 AD): Replace the uncertain life length with expected.

De Witt (17th century): Principle of equivalence:
\[
\pi = \mathbb{E} \left[ b \, e^{-\int_0^T r_u \, du} \, 1[T_x > T] \right] = b \, e^{-\int_0^T (r_u + \mu_{x+u}) \, du}
\]
The contract is long term and the development of $r$ and $\mu$ is uncertain. The principle of equivalence must be recast as

$$\pi = \mathbb{E} \left[ b e^{-\int_0^T r_u \, du} 1[T_x > T] \mid F_t \right] = b e^{-\int_0^T (r_u + \mu x + u) \, du}$$

A resolution is *perfect index-linked insurance*:

$$b = \pi e^{\int_0^T (r_u + \mu x + u) \, du}$$

Another possibility is *with profit insurance*.

NB! Solvency with probability 1 (if implemented properly). No model assumptions about $r$ and $\mu$.

A different approach is Alternative Risk Transfer. Securitization merits discussion.
Market methods through securitization. Model: 
$r$ constant

\[ \mu_{x+u} = m_{x+u} + \sigma X_i, \quad i - 1 \leq u < i, \quad i = 1, \ldots, T \]

(Physical) Probability measure $\mathbb{P}$: the $X_i$ i.i.d. $\text{Bin}(0, p)$,

Introduce mortality derivative with payment $G(X_1, \ldots, X_T)$ at time $T$. Hedge the claim

\[ H(X_1, \ldots, X_T) = e^{-\int_0^T m_{x+u} du} - \sigma \sum_{i=1}^T X_i \]

perfectly by trading in the bank account and the derivative (the same maths as for the financial claim seen before).

And we are done!
Second thoughts:

How to determine $Q$? Is the model right?
If $H = G$ things are simple, no model needed. If not, there is superimposed model risk.
Dangers are looming: Speculation, Asymmetry of information, Bubbles and panic, ...
Does the product make sense to customers?
Does the very idea make economic sense?
Nothing wrong with with profit. The problem was that it was not implemented properly. Presumably, under the pressure of competition, the insurers weren’t sufficiently prudent in their assumptions about future interest and mortality developments (technical basis), the surpluses were squandered on premature bonuses, and bonuses were even guaranteed before the surpluses were gained. The blame should not be put on the with profit concept, it should be placed at the door of the actuaries. (If you get a ticket for speeding, you should not blame your Mercedes.)

Analysis of interest, mortality, expenses and other collective risk factors is an essential part of actuarial work, but an even more important part - that only the actuary can do - is the design of the scheme and the insurance contracts.
THE MATHEMATICS OF RISK SHARING
Consider multi-state life insurance contract with states \{0, \ldots, n\}, starting in state 0 at time 0 and terminating at time \(T\). State of policy at time \(t\) is \(Z(t)\).

Total payments (benefits less premiums) in time interval \([0, t]\) are \(B(t)\) given by

\[
dB(t) = \sum_j I_j(t-)dB_j(t) + \sum_{j \neq k} b_{jk}(t) dN_{jk}(t) \quad t \in [0, T],
\]

\(I_j(t) = 1[Z(t) = j]\) indicator processes,
\(N_{jk}(t) = \#\{s \in (0, t); Z(s-) = j, Z(s) = k\}\) counting processes,
\(B_j\) is payment stream (general annuity) running in state \(j\),
\(b_{jk}\) is sum assured payable upon transition from state \(j\) to state \(k\).

Filtration generated by the policy is \(\mathbb{F}^N = (\mathcal{F}_t^N)_{t \in [0, T]}\).
Payments are deposited/withdrawn from investment portfolio with interest rate $r(t)$ at time $t$.
Transition intensities for the policy are $\mu_{jk}(t)$, $j \neq k$.

$Y(t) = (r(t), \mu_{jk}(t), j \neq k)$ is stochastic process generating filtration $F^Y = (\mathcal{F}^Y_t)_{t \in [0,T]}$.

Thus, $N$ is Markov given $Y$. 

PRINCIPLE OF EQUIVALENCE requires balance on the average in an infinitely large portfolio. Future interest and mortality are unknown at time 0. Therefore, equivalence must mean

$$
E \left[ \int_{0-}^{T} e^{-\int_{0}^{\tau} r \, dB(\tau)} \left| \mathcal{F}_{T}^{Y} \right. \right] = 0
$$

or

$$
\int_{0-}^{T} e^{-\int_{0}^{t} r \sum_{j} p_{0j}(0,t) \left( dB_{j}(t) + \sum_{k;k \neq j} b_{jk}(t) \mu_{jk}(t) \, dt \right)} = 0 \quad (5)
$$

with probability one. Therefore, $B$ must be adapted to $\mathcal{F}_{Y} \lor \mathcal{F}_{N}$. 

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INDEX-LINKED INSURANCE

Let $r^*$ and $\mu^*_{jk}$, $j \neq k$, be base-line interest and transition rates representing best estimates at time 0. 
Stipulate base-line payments $B_j^*$ and $b_{jk}^*$ that would be suitable under this scenario and satisfying equivalence:

$$\int_{[0,T]} e^{-\int_0^t r^*(s) \, ds} \left( \sum_j p_{0j}^*(0, t) \, dB_j^*(t) + \sum_{j \neq k} p_{0j}^*(0, t) \, \mu_{jk}^*(t) \, b_{jk}^*(t) \, dt \right) = 0. \tag{6}$$
Clear-cut index-linked policy has state-wise annuities $B_j$ and assurances $b_{jk}$ given by

$$dB_j(t) = \frac{e^{-\int_0^t r^*(0,t)p_{0j}(0,t)}}{e^{-\int_0^t r p_{0j}(0,t)}} dB_j^*(t)$$

$$b_{jk}(t) = \frac{e^{-\int_0^t r^* p_{0j}(0,t) \mu_{jk}(t)}}{e^{-\int_0^t r p_{0j}(0,t) \mu_{jk}(t)}} b_{jk}^*(t)$$

Inserting this into (5) gives (6), hence equivalence with probability 1.

**NO ASSUMPTIONS ABOUT ENVIRONMENT PROCESSES.** By conditioning on the factual outcome of the environment process, nothing depends on its distribution, which cannot be reliably modelled for the long term anyway.
MORAL: Do not let vital decisions that determines the survival of an insurance scheme depend on speculative assumptions about interest, mortality, etc over the next century.
WITH PROFIT: Payments $B_j^*$ and $b_{jk}^*$ guaranteed at time 0. Designed by equivalence principle using prudent technical basis with elements $r^*$ and $\mu_{jk}^*$. Surpluses that emerge are paid back as bonuses, cash dividends or additional benefits:

$D(t)$ total of dividends paid out cash by time $t$
$Q(t)$ total additional units of additional benefits $B^{*+}$ guaranteed by time $t$.

At time $t$ the company promises to pay $Q(t) (B^{*+}(\tau) - B^{*+}(t))$ for $\tau \in (t, T]$.

$D$ and $Q$ must be non-decreasing, and $D(0) = Q(0) = 0$.

Payments from the company to the insured are

$$dB(t) = dB^*(t) + Q(t-)dB^{*+}(t) + dD(t)$$
$B^*$, $B^{*+}$ are of the standard form with $B^*_j$, $b^*_{jk}$ deterministic. $Q$ and $D$ are adapted to $\mathbb{F}^N \vee \mathbb{F}^Y$. They are not stipulated in the contract, but controlled by the company in view of the past experience and with a view to customer needs and solvency. Liability in respect of future payments at time $t$ is

$$V_{Z(t)}^*(t) + Q(t) V_{Z(t)}^{*+}(t)$$

Discounted surplus at time $t$ is

$$\tilde{W}(t) = -\int_0^t e^{-\int_0^\tau r \left( dB^*(\tau) + Q(\tau-) dB^{*+}(\tau) + dD(\tau) \right)} - e^{-\int_0^t r \left( V_{Z(t)}^*(t) + Q(t) V_{Z(t)}^{*+}(t) \right)}$$

(7)

$$\tilde{W}(0) = -\Delta B^*(0) - V_0^*(0) = 0$$
$$\tilde{W}(T) = -\int_0^T e^{-\int_0^\tau r} \left( dB^*(\tau) + Q(\tau-) dB^{*+}(\tau) + dD(\tau) \right)$$

If first order basis is chosen on the entirely safe side and bonuses are allotted with sufficient prudence, then one can arrange that \( \tilde{W}(t) \geq 0 \) for all \( t \), and there is no solvency problem.

Solvency requirement:

$$\mathbb{E}\left[ \tilde{W}(t) \mid \mathcal{F}_t^Y \right] \geq 0, \ t \in [0, T]$$

Equivalence:

$$\mathbb{E}\left[ \tilde{W}(T) \mid \mathcal{F}_T^Y \right] = 0$$
Applying Itô to $\tilde{W}(t)$:

$$d\tilde{W}(t) = e^{-\int_0^t r \left( dC(t) - dD(t) - dQ(t)V_{Z(t)}^+(t) + dM^*(t) \right)}$$

$C(t)$ is drift term representing "technical surplus":

$$dC(t) = (r(t) - r^*(t)) \left( V_{Z(t)}^*(t) + Q(t)V_{Z(t)}^+(t) \right) dt$$

$$+ \sum_{k; k \neq Z(t)} \left( \mu_{Z(t)}^*(k(t)) - \mu_{Z(t)} k(t) \right) \left( R_{Z(t)}^*(k(t)) + Q(t)R_{Z(t)}^+(k(t)) \right) dt$$

$$R_{jk}^*(t) = b_{jk}^*(t) + V_k^*(t) - V_j^*(t), \quad R_{jk}^{*+}(t) = b_{jk}^{*+}(t) + V_k^{*+}(t) - V_j^{*+}(t)$$

$M^*(t)$ is martingale representing purely individual life history randomness:

$$dM^*(t) = \sum_{g \neq h} \left( R_{jk}^*(t) + Q(t-)R_{jk}^{*+}(t) \right) \left( dN_{jk}(t) - I_{j}(t)\mu_{jk}(t) dt \right)$$
Writing $\tilde{W}(T) = \int_{[0,T]} d\tilde{W}(\tau)$, and forming conditional expectation, equivalence can be recast as

$$\mathbb{E} \left[ \int_0^T e^{-\int_0^\tau r \left( dC(\tau) - dD(\tau) - dQ(t)V_{Z(t)}^*(t) \right)} \bigg| \mathcal{F}_T^Y \right] = 0$$

NO ASSUMPTIONS ABOUT ENVIRONMENT PROCESSES!
REFERENCES:
Resumé: A sustained trend of improving longevity prospects and repeated down-falls of the financial markets have been identified as the principal causes of the pension crisis, ongoing since more than a decade now. This perception needs to be completed by a broader historical view: already about a century ago insurance mathematics provided principles for perfect management of demographic and financial risk through so-called risk sharing in the form of with-profit schemes designed as follows: Upon the inception of the contract the premiums and the benefits are guaranteed in nominal amounts for the entire contract period; Premiums are determined by the principle of equivalence (expected balance between discounted premiums and discounted benefits) based on a worst-case economic-demographic scenario called the first order (technical) basis; The systematic surpluses generated by the conservative premiums are redistributed to
the insured in arrears as bonus. However simple the idea, its technical implementation is non-trivial since the scheme needs to be, not only equitable, but also fair in the following precise actuarial sense: with probability 1 the principle of equivalence should ultimately be attained conditionally, given the course of economic-demographic events over the contract period.

The purpose of the present paper is to pursue the idea of risk sharing in life insurance and develop it further in two specific directions: (1) Revisit the with-profit concept and reshape it in terms of the cash flow dynamics in a manner that, firstly, comprises in a unified manner the various forms of bonuses (cash dividends, terminal bonus, and guaranteed added benefits) and, secondly, allows the first order reserve to be based on technical elements that are dynamically adapted
to experience (“prudent predictions”). (2) Work out in full generality what can be called perfectly index-linked insurance, whereby payments (premiums as well as benefits) are linked to financial and demographic indices in a manner that automatically establishes solvency and fairness as defined above.